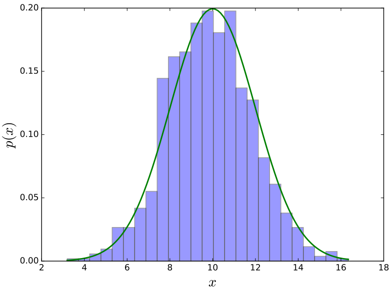
Properties of power-law distributions

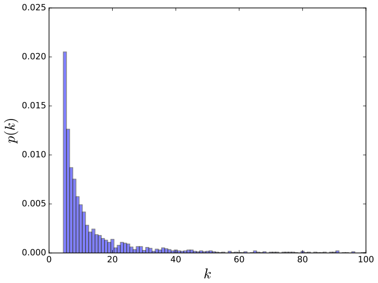
Snehal M Shekatkar

Peaked distributions



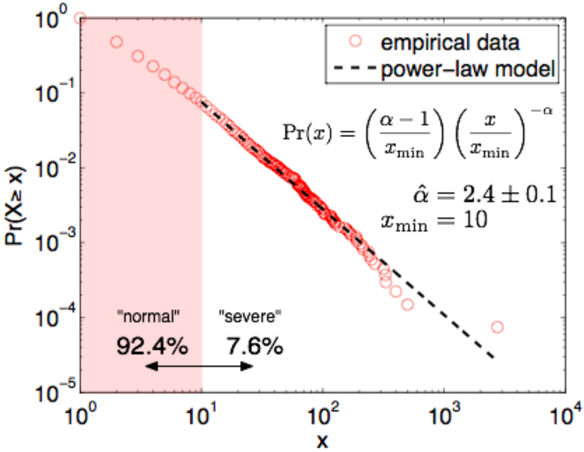
- Sizes of apples
- Heights of humans
- Number of goals in a football match
- Error sizes in measurements
- Blood pressures of humans
- Marks obtained in a test
- Daily temperatures in a given region
- Number of people in a family

Right-skewed distributions



- Frequencies of words (Zipf's law)
- Numbers of citations
- Web hits
- Copies of books sold
- Magnitude of earthquakes
- Diameters of moon craters
- Intensity of wars
- Amount of wealth
- Frequencies of family names
- City populations
- Sizes of computer files

Severity of terrorist attacks



(Taken from the blog by Aaron Clauset)

Normalization for the pure power-law

- $p_k = Ck^{-\alpha} \quad \text{with} \quad p_0 = 0$
- Normalization condition: $\sum_{k=0}^{\infty} p_k = 1$
- $C = \frac{1}{\sum_{k=0}^{\infty} k^{-\alpha}} = \frac{1}{\zeta(\alpha)} \implies p_k = \frac{k^{-\alpha}}{\zeta(\alpha)}$

Normalization for non-pure power-laws

- Normalization over the tail $p_k = \frac{k^{-\alpha}}{\sum_{k=k_{\min}}^{\infty} k^{-\alpha}} = \frac{k^{-\alpha}}{\zeta(\alpha, k_{\min})}$

- $C \approx \frac{1}{\int\limits_{k_{\min}}^{\infty} k^{-\alpha} dk} = (\alpha - 1) k_{\min}^{\alpha-1} \Longrightarrow p_k \approx \frac{\alpha-1}{k_{\min}} \left(\frac{k}{k_{\min}} \right)^{-\alpha}$

- $P_k = \left(\frac{k}{k_{\min}} \right)^{-(\alpha-1)}$

Moments

- $\langle k \rangle = \sum_{k=0}^{\infty} k p_k \quad , \quad \langle k^2 \rangle = \sum_{k=0}^{\infty} k^2 p_k \quad , \quad \langle k^m \rangle = \sum_{k=0}^{\infty} k^m p_k$
- Non-pure power-law:

$$\langle k^m \rangle = \sum_{k=0}^{k_{\min}-1} k^m p_k + C \sum_{k=k_{\min}}^{\infty} k^{m-\alpha}$$

$$\langle k^m \rangle \approx \sum_{k=0}^{k_{\min}-1} k^m p_k + C \int_{k_{\min}}^{\infty} k^{m-\alpha} dk$$

$$= \sum_{k=0}^{k_{\min}-1} k^m p_k + \frac{C}{m-\alpha+1} \left[k^{m-\alpha+1} \right]_{k_{\min}}^{\infty}$$

For finite value, $m - \alpha + 1 < 0 \implies \alpha > m + 1$
$\implies$ Whenever $\alpha < 3$, $\langle k^2 \rangle$ diverges!

Moments for real networks

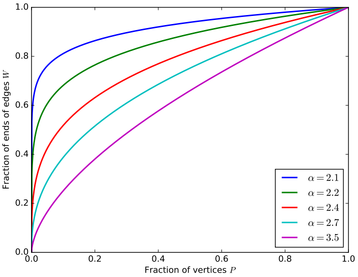
- Since n is finite, all moments are finite

$$\langle k^m \rangle = \frac{1}{n} \sum_{i=1}^{n} k_i^m$$

- Usually theoretical infinite corresponds to a very large values in finite networks

- For the Internet, $\langle k^2 \rangle = 1159 >> \langle k \rangle^2 = 25$

Lorenz curves



- Fraction of edges connected to vertcies with highest degrees:

$$W = P^{(\alpha-2)/(\alpha-1)}$$

- For the World Wide Web ($\alpha = 2.2$), with $P = 0.5$, we get $W \approx 0.9$
- With $W = 0.5$, we get $P = 0.015$
- Similar results exist for citation networks, Internet etc even in the absense of pure-power law

Validation of power-law hypothesis

PERSPECTIVE

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Albert-László Barabási

For decades, we tacitly assumed that the components of such complex systems as the cell, the society, or the Internet are randomly wired together. In the past decade, an avalanche of research has shown that many real networks, independent of their age, function, and scope, converge to similar architectures, a universality that allowed researchers from different disciplines to embrace network theory as a common paradigm. The decade-old discovery of scale-free networks was one of those events that had helped catalyze the emergence of network science, a new research field with its distinct set of challenges and accomplishments.

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ARTICLE

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Scale-free networks are rare

Anna D. Broido[1] & Aaron Clauset[2,3,4]

Real-world networks are often claimed to be scale free, meaning that the fraction of nodes with degree k follows a power law $k^{-\alpha}$, a pattern with broad implications for the structure and dynamics of complex systems. However, the universality of scale-free networks remains controversial. Here, we organize different definitions of scale-free networks and construct a severe test of their empirical prevalence using state-of-the-art statistical tools applied to nearly 1000 social, biological, technological, transportation, and information networks. Across these networks, we find robust evidence that strongly scale-free structure is empirically rare, while for most networks, log-normal distributions fit the data as well or better than power laws. Furthermore, social networks are at best weakly scale free, while a handful of technological and biological networks appear strongly scale free. These findings highlight the structural diversity of real-world networks and the need for new theoretical explanations of these non-scale-free patterns.

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Link to the Twitter War

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- Power-Law Distributions in Empirical Data, A. Clauset, C. Shalizi, M.E.J. Newman, SIAM Review (2009)