Erdős-Rényi model

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Two types of network models

- **Random Graph Models**: Can we construct graphs which mimic the real-world networks?

- **Evolutionary Network Models**: Why the real-world networks possess the properties they do?

$G(n, m)$

- Take n vertices and m edges
- Distribute m edges among the $\binom{n}{2}$ vertex pairs randomly
- The $G(n, m)$ is the ensemble such that every graph appears with equal probability $P(G) = \frac{1}{\binom{\binom{n}{2}}{m}}$
- The average properties can be calculated by averaging over the ensemble: $\langle l \rangle = \sum_G P(G) l(G)$
- The model is not flexible enough

Erdős-Rényi model $G(n,p)$

- Paul Erdős and Alfred Rényi proposed a model in early sixtees which marked the start of the random graph theory
- Take n vertices, and connect every pair of vertices with probability p
- A generated graph can have any number of edges between 0 and $\binom{n}{2}$
- Each graph in the ensemble occurs with probability:

$$P(G) = p^m (1-p)^{\binom{n}{2}-m}$$

Properties of $G(n, p)$

- Average number of edges : $\binom{n}{2}p$
- Average degree : $c = p(n-1)$

Degree distribution of the ER graph

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

As $n \to \infty$, we have,

$$(1-p)^{n-1-k} = \left(1 - \frac{c}{n-1}\right)^{n-1-k} = e^{-c}$$

And also,

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!k!} \approx \frac{(n-1)^k}{k!}$$

Combining,

$$p_k = \frac{(n-1)^k}{k!} p^k e^{-c} = \frac{(n-1)^k}{k!} \left(\frac{c}{n-1}\right)^k e^{-c} = e^{-c}\frac{c^k}{k!}$$

Poisson Random Graph!

Clustering coefficient

Any two vertices have the same connection probability. Hence,

$$C = \frac{c}{n-1}$$

For sparse graphs (i.e. with constant c), $C \to 0$